

# Mixed-integer Quadratic Programming is in NP

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## Abstract

*Mixed-integer quadratic programming (MIQP)* is the problem of optimizing a quadratic function over points in a polyhedral set where some of the components are restricted to be integral. In this paper, we prove that the decision version of mixed-integer quadratic programming is in NP, thereby showing that it is NP-complete. This is established by showing that if the decision version of mixed-integer quadratic programming is feasible, then there exists a solution of polynomial size. This result generalizes and unifies classical results that quadratic programming is in NP [11] and integer linear programming is in NP [1, 12, 4, 9].

## 1 Introduction

*Mixed-integer quadratic programming (MIQP)* is the problem of optimizing a quadratic function over points in a polyhedral set that have some components integer, and others continuous. More formally, a MIQP problem is an optimization problem of the form:

$$\begin{aligned} \min \quad & x^\top Hx + c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}, \end{aligned} \tag{1}$$

where  $H \in \mathbb{Q}^{n \times n}$  and is symmetric,  $c \in \mathbb{Q}^n$ ,  $A \in \mathbb{Q}^{m \times n}$  and  $b \in \mathbb{Q}^m$ . The decision version of this problem is: Does there exist a feasible solution to  $\mathcal{F}(H, c, d, A, b)$  where  $\mathcal{F}(H, c, d, A, b)$  is the set of  $x$  satisfying

$$\begin{aligned} & x^\top Hx + c^\top x + d \leq 0 \\ & x \in \mathcal{C} := \{x : Ax \leq b\} \\ & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}. \end{aligned} \tag{2}$$

The special case of MIQP when all variables are required to be integer ( $p = n$ ) is called *integer quadratic programming (IQP)*. It is well known that IQP is NP-hard. We show that the decision versions of IQP and MIQP lie in NP. Therefore, decision version of IQP and MIQP are NP-complete. This result generalizes and unifies classical results that *quadratic programming* is in NP [11] and *integer linear programming* is in NP [1, 12, 4, 9].

Recently, Del Pia and Weismantel [2] showed that IQP can be solved in polynomial time when  $n = 2$ . It is a major open question whether IQP can be solved in polynomial-time for fixed dimension.

## 1.1 Statement of result and discussion

Given a rational vector/matrix, its *complexity* is the bit-size of its smallest binary encoding. The complexity of a rational polyhedron  $P \subseteq \mathbb{R}^n$  is the smallest complexity of a matrix  $[A \ b]$  such that  $P = \{x : Ax \leq b\}$ . The complexity of a finite set of objects is the sum over the complexity of the constituents objects. We will prove the following result.

**Theorem 1.** *Let  $n, p \in \mathbb{Z}_{++}$ . Let  $H \in \mathbb{Q}^{n \times n}$ ,  $c \in \mathbb{Q}^n$ ,  $d \in \mathbb{Q}$ ,  $A \in \mathbb{Q}^{m \times n}$ ,  $b \in \mathbb{Q}^m$ , and let  $\phi$  be the complexity of  $\{H, c, d, A, b\}$ . If  $\mathcal{F}(H, c, d, A, b)$  is non-empty, then there exists  $x^0 \in \mathcal{F}(H, c, d, A, b)$  such that the complexity of  $x^0$  is bounded from above by  $f(\phi)$  where  $f$  is a polynomial function.*

Theorem 1 directly implies the following.

**Corollary 2.** *The decision versions of IQP and MIQP are NP-complete.*

*Proof.* Given a graph  $G = (V, E)$  and an integer  $k$ , determining whether there is a cut of cardinality at least  $k$  in  $G$  is NP-complete [3, 5]. It is well known that such problem can be written as the decision IQP problem

$$\begin{aligned} \sum_{v_i v_j \in E} (x_i + x_j - 2x_i x_j) &\geq k \\ x_i &\in \{0, 1\}^n \quad \forall v_i \in V. \end{aligned}$$

Hence any problem in NP can be polynomially transformed to a decision IQP. Theorem 1 proves that there is a polynomial-length certificate for yes-instances of decision MIQP, showing that decision IQP and MIQP are in NP.  $\square$   $\square$

We end this section by contrasting the result of Theorem 1 with several well-known negative results when one considers a more general version of decision IQP by varying the number of quadratic inequalities.

1. ‘Many’ general quadratic inequalities: By using a simple reduction from the problem of determining the feasibility of a quartic equation in 58 non-negative integer variables, we obtain that determining the feasibility of a system with  $2 \left( \binom{58}{2} + 58 + 1 \right)$  quadratic inequalities and 58 linear inequalities in  $\left( \binom{58}{2} + 58 \right)$  continuous variables and 58 integer variables is undecidable (see Theorem 3.2 and Theorem 3.3(i) in [7]). Therefore already with 3424 quadratic inequalities, 58 linear inequalities, 58 integer variables, 1711 continuous variables, it is not possible to bound the size of smallest feasible solution.
2. Two general quadratic inequalities: In the presence of two quadratic inequalities (in fact one quadratic equation) in two variables, there exist examples (the so called *Pellian* and *anti-Pellian* equations) where the minimal binary encoding length of any feasible integral solution is exponential in the complexity of the instance [8].
3. ‘Many’ convex quadratic inequalities: Consider the following system of inequalities [6]:

$$x_1 \geq 2 \tag{3}$$

$$x_i \geq x_{i-1}^2 \quad \forall i \in \{2, \dots, n\} \tag{4}$$

$$x \in \mathbb{Z}^n. \tag{5}$$

It is clear that for this system, the minimal binary encoding length of any feasible integral solution is exponential in the complexity of the instance.

The above examples serve to highlight the fact that the result of Theorem 1 (a feasible system of inequalities with exactly one quadratic inequality always has at least one integer feasible solution of small size) is tight with respect to the number of quadratic inequalities.

The rest of the paper is organized as follows. Section 2 collects all notation and results that are needed to prove Theorem 1. Section 3 presents a proof of Theorem 1.

## 2 Preliminaries

### 2.1 Notation

Throughout this paper, we use  $e^i$  to represent the  $i$ -th vector of the standard basis of  $\mathbb{R}^n$ ,  $\text{sign}(u)$  to represent the sign of a real number  $u$ ,  $\dim(S)$  to represent the affine dimension of  $S$ ,  $\text{conv}(S)$  to represent the convex hull of a set  $S$ ,  $\text{cone}(S)$  to represent the conic hull of a set  $S$ , and  $\text{int.cone}(S)$  to represent the set  $\sum_{r^j \in S} \lambda_j r^j, \lambda_j \in \mathbb{Z}_+ \forall j$ .

Given an object  $\mathcal{O}$  and another object  $f(\mathcal{O})$  that is a function of it, we say that  $f(\mathcal{O})$  has  $\mathcal{O}$ -small complexity if the complexity of  $f(\mathcal{O})$  is at most a polynomial function of the complexity of  $\mathcal{O}$  (or more precisely, there is a polynomial  $p$  such that for every input object  $\mathcal{O}$ , the complexity of  $f(\mathcal{O})$  is at most  $p(\text{complexity}(\mathcal{O}))$ ).

### 2.2 Quadratic programming is in NP

*Quadratic programming (QP)* is the special case of MIQP when all variables are continuous ( $p = 0$ ). Vavasis [11] proved that the decision version of QP is in NP.

**Theorem 3.** *The feasibility problem over the continuous relaxation of (2) is in NP. Moreover, suppose that the continuous relaxation of (1) has a global optimal solution. Then there exists a system of rational linear equations of  $\{H, c, d, A, b\}$ -small complexity whose solution is one of the global optimal solutions.*

### 2.3 Mixed-integer linear programming is in NP

We will need the following generalization of a classical result that can be used to prove that the decision version of mixed integer linear programming (MIP) is in NP. We say that a pointed polyhedral cone  $C \subseteq \mathbb{R}^n$  is a *simple cone* if the number of extreme rays is equal to the dimension of the cone.

**Proposition 4.** *Let  $P = \{x : Ax \leq b\} \subseteq \mathbb{R}^{p+q}$  be a rational pointed polyhedron. Then there is a finite family  $\{P_i\}_i$  of polytopes, and a finite family  $\{R_K\}_{K \in \mathcal{K}}$  of subsets of extreme rays of  $P$  with the following properties:*

1.  $P \cap (\mathbb{Z}^p \times \mathbb{R}^q) = \bigcup_{i, K \in \mathcal{K}} (P_i + \text{int.cone}(R_K))$ ;
2. Each polytope  $P_i$  and each vector in  $R_K$  has  $[A \ b]$ -small complexity;
3. For each  $K \in \mathcal{K}$ , all vectors in  $R_K$  are integral;
4. Each cone  $\text{cone}(R_K)$  is simple.

*Proof.* Assume  $P$  is non-empty, otherwise there is nothing to prove. By standard polyhedral theory, there is a set of vectors  $\{v^1, \dots, v^\ell\}$  (the vertices of  $P$ ) and a set of *integral* vectors  $\{r^1, \dots, r^m\}$  (a scaling of the the extreme rays of  $P$ ) such that  $P = \text{conv}\{v^1, \dots, v^\ell\} + \text{cone}\{r^1, \dots, r^m\}$  and the  $v^i$ 's and  $r^j$ 's have  $[A \ b]$ -small complexity (see for example Chapters 7 and 10 in [10]).

For a subset  $K \subseteq \{1, \dots, m\}$ , let  $R_K = \{r^j : j \in K\}$  be the set of extreme rays indexed by  $K$ . Let  $\mathcal{K}$  be the set of  $K$ 's such that the cone  $\text{cone}(R_K)$  is simple. Finally consider the set  $B = \bigcup_{K \in \mathcal{K}} B^K$  where

$$B^K = \left\{ x \in \mathbb{Z}^p \times \mathbb{R}^q : x = v + \sum_{j \in K} \mu_j r^j, \ v \in \text{conv}\{v^1, \dots, v^\ell\}, \ \mu_j \in [0, 1] \ \forall j \right\}. \quad (6)$$

Notice that  $B$  is a union of the polytopes given by the fibers  $\{\bar{y}\} \times \{z \in \mathbb{R}^q : (\bar{y}, z) \in B^K\}$  ranging over all  $K \in \mathcal{K}$  and  $\bar{y} \in B^K|_p$ , where  $B^K|_p$  is the projection of  $B^K$  to the first  $p$  coordinates. Using the fact that  $v^i$ 's and  $r^j$ 's have  $[A \ b]$ -small complexity and  $|K| \leq p + q$ , we get that all points in  $B^K|_p$  have  $[A \ b]$ -small complexity. Hence each of these fibers also has  $[A \ b]$ -small complexity since it is the intersection of two  $[A \ b]$ -small complexity polyhedron  $\{x \in \mathbb{R}^p \times \mathbb{R}^q : x = v + \sum_{j \in K} \mu_j r^j, \ v \in \text{conv}\{v^1, \dots, v^\ell\}, \ \mu_j \in [0, 1] \ \forall j\}$  and  $\{x \in \mathbb{R}^p \times \mathbb{R}^q : x|_p = \bar{y}\}$ . Let  $\{P_i\}_i$  be this collection of fibers.

By construction, properties 2, 3 and 4 of the proposition are satisfied, so it suffices to show property 1. By using distributivity of union and Minkowski sums, notice  $\bigcup_{i, K \in \mathcal{K}} (P_i + \text{int. cone}(R_K)) = B + \bigcup_{K \in \mathcal{K}} \text{int. cone}(R_K)$ .

To show  $P \cap (\mathbb{Z}^p \times \mathbb{R}^q) \subseteq B + \bigcup_{K \in \mathcal{K}} \text{int. cone}(R_K)$ , take a point  $x \in P \cap (\mathbb{Z}^p \times \mathbb{R}^q)$ . We can write it as  $x = v + r$  for  $v \in \text{conv}(v^i)_i$  and  $r \in \text{cone}(r^j)_j$ . Using Carathéodory's Theorem, we see that there exists  $K \in \mathcal{K}$  such that the simple cone  $\text{cone}(R_K)$  contains  $r$ . Then consider multipliers  $\mu_j \in \mathbb{R}_+$  for  $j \in K$  such that  $r = \sum_{j \in K} \mu_j r^j$ . Breaking up the multipliers into their fractional and integer parts, we get that

$$x = v + \sum_{j \in K} (\mu_j - \lfloor \mu_j \rfloor) r^j + \sum_{j \in K} \lfloor \mu_j \rfloor r^j.$$

Clearly the last term belongs to  $\text{int. cone}(R_K)$ . Moreover, this term is integer (since the  $r^j$ 's are integer) and  $x \in \mathbb{Z}^p \times \mathbb{R}^q$ , thus the remaining part  $v + \sum_{j \in K} (\mu_j - \lfloor \mu_j \rfloor) r^j = x - \sum_{j \in K} \lfloor \mu_j \rfloor r^j$  belongs to  $\mathbb{Z}^p \times \mathbb{R}^q$  and hence to  $B$ . Thus  $x \in B + \text{int. cone}(R_K)$ , concluding this part of the proof.

We now show the reverse direction  $P \cap (\mathbb{Z}^p \times \mathbb{R}^q) \supseteq B + \bigcup_{K \in \mathcal{K}} \text{int. cone}(R_K)$ . It is easy to see that  $P \supseteq B + \bigcup_{K \in \mathcal{K}} \text{int. cone}(R_K)$ , since  $P = \text{conv}(v^i)_i + \text{cone}(r^j)_j$  and  $B \subseteq \text{conv}(v^i)_i + \text{cone}(r^j)_j$  and  $\text{int. cone}(R_K) \subseteq \text{cone}(r^j)_j$ . Also,  $B \subseteq \mathbb{Z}^p \times \mathbb{R}^q$  and  $\text{int. cone}(R_K) \subseteq \mathbb{Z}^{p+q}$  (again since the  $r^j$ 's are integral), and hence  $B + \bigcup_{K \in \mathcal{K}} \text{int. cone}(R_K) \subseteq \mathbb{Z}^p \times \mathbb{R}^q$ . This concludes the proof.  $\square$

One way of interpreting this decomposition is the following: Notice that each set  $\text{int. cone}(R_K)$ , for  $K \in \mathcal{K}$ , is linearly isomorphic to  $\mathbb{Z}_+^{|K|}$ ; this proposition then asserts that we can decompose any mixed-integer linear set into (overlapping) sets that are affinely isomorphic to some  $\mathbb{Z}_+^{n'}$ . Resorting to the product structure of  $\mathbb{Z}_+^{n'}$  will be instrumental in our main proof.

Also, notice that this proposition proves that the decision version of MIP is in NP:

**Proposition 5.** *The decision version of MIP is in NP.*

Indeed, if the decision version of a MIP is true, then one can present a vertex of one of the polytopes  $P_i$  as a certificate.

### 3 Proof of Theorem 1

We begin this section with some technical lemmas.

**Lemma 6** (Normalizing hyperplane). *Let  $C = \text{cone}\{r^1, \dots, r^s\} \subseteq \mathbb{R}^n$  be a pointed cone. Then there exists a hyperplane  $\mathcal{H} = \{x : f^\top x = 1\}$  such that:*

1. *The complexity of  $f$  is polynomially bounded by the maximum complexity of  $r^i$ , for  $i = 1, \dots, s$ ;*
2. *If  $x \in C$  and  $\|x\| = 1$ , then  $f^\top x \geq \frac{1}{R}$  where  $R = \max_{i \in \{1, \dots, s\}} \{\|r^i\|\}$ .*
3.  *$C \cap \mathcal{H}$  is bounded.*

*Proof.* Let  $\{r^{s+1}, \dots, r^t\}$  be a minimal subset of  $\{e^1, \dots, e^n\}$  with the property that the cone  $C' = \text{cone}\{r^1, \dots, r^s, r^{s+1}, \dots, r^t\}$  is full-dimensional. Clearly  $|\{r^{s+1}, \dots, r^t\}| = n - d$ , where  $d$  is the dimension of  $C$ , and  $C'$  is pointed. Let  $f$  be an extreme point of the following polyhedron:  $\{w : w^\top r^i \geq 1 \ \forall i \in \{1, \dots, t\}\}$  (an extreme point exists since the rank of the matrix defining the polyhedron is  $n$ ). Then  $f$  satisfies the first and third criteria. Suppose  $\hat{x} \in C$  and  $\|\hat{x}\| = 1$ . There exists  $0 < \mu \leq 1$  such that  $\mu\hat{x}$  belongs to the polytope defined as the convex hull of  $\{\frac{r^i}{\|r^i\|} : i = 1, \dots, s\}$  (since the maximum norm of any vector in this polytope is 1). Thus there exist  $\lambda_i$ ,  $i = 1, \dots, s$ , with  $\sum_{i=1}^s \lambda_i = 1$  such that  $f^\top \hat{x} \geq f^\top (\mu\hat{x}) = \sum_{i=1}^s \lambda_i \frac{1}{\|r^i\|} f^\top r^i \geq \frac{1}{R}$ .  $\square$

Sometimes we will apply Lemma 6 to a pointed cone  $C$ , without giving explicitly the set of rays  $\{r^1, \dots, r^s\}$ . It is well known (see for example Theorem 10.2 in [10]) that facet and vertex complexity of a rational polyhedron are polynomially related. Hence there exist vectors  $r^1, \dots, r^s$ , each of  $C$ -small complexity, such that  $C = \text{cone}\{r^1, \dots, r^s\}$ . Hence, in this case, Lemma 6 implies that there exists a normalizing hyperplane  $\mathcal{H} = \{x : f^\top x = 1\}$  such that:

1.  $f$  has  $C$ -small complexity;
2. For every nonzero  $x \in C$ , there exists  $\mu > 0$  such that  $\mu x \in \mathcal{H}$ .

The following lemma outlines a crucial decomposition strategy for searching integer feasible points.

**Lemma 7.** *Let  $C$  be a simple pointed cone such that  $x^\top Hx \geq 0$  for every  $x \in C$ . Let  $\mathcal{H} = \{x : f^\top x = 1\}$  be the normalizing hyperplane from Lemma 6. Then there exists a finite family of simple cones  $C^i$ ,  $i \in I$  such that*

- (a)  $\bigcup_{i \in I} C^i = C$ ,
- (b) *for every  $i \in I$ , if a face  $F$  of  $C^i$  satisfies  $\min\{x^\top Hx : x \in F \cap \mathcal{H}\} = 0$ , then there exists an extreme ray  $v$  of  $F$  with  $v^\top Hv = 0$ ,*
- (c) *for every  $i \in I$ ,  $C^i$  has  $\{H, C\}$ -small complexity and dimension of  $C^i$  is equal to the dimension of  $C$ .*

*Proof.* The proof is by induction on the dimension  $n$  of the cone. If the cone has dimension one, then the claim is trivially true.

Since  $C \cap \mathcal{H}$  is a compact convex set, by Theorem 3 there exists an optimal solution  $\bar{x}$  of the problem  $\min\{x^\top Hx : x \in C \cap \mathcal{H}\}$  that has  $\{H, C, \mathcal{H}\}$ -small complexity. As  $\mathcal{H}$  has  $C$ -small complexity,  $\bar{x}$  has  $\{H, C\}$ -small complexity. If the minimum value is strictly positive, then the result is trivially true. So we now assume that the minimum value is zero.

Let  $F_i, i \in I$  be the facets of  $C$  that do not contain  $\bar{x}$ . By induction, for every  $i \in I$ , there exist finitely many simple cones (of dimension  $n - 1$ , and with  $n - 1$  extreme rays)  $C_i^j, j \in J(i)$  that satisfy (a) and (b) with respect to the  $n - 1$  dimensional cone  $F_i$ . We show that the family of cones

$$\{\text{cone}\{C_i^j \cup \{\bar{x}\}\} : i \in I, j \in J(i)\} \quad (7)$$

satisfies (a) and (b). Since for every  $i \in I$ , the vector  $\bar{x}$  is affinely independent from all the vectors in  $F_i$ , each element of (7) is a simple cone. It is straightforward to verify that (a) holds. Condition (b) holds by induction for all the faces of  $C_i^j, i \in I, j \in J(i)$ , and it holds also for all the remaining faces of the cones in (7) because they all contain  $\bar{x}$  as an extreme ray.

The above proof of (a) and (b) can be seen as a constructive algorithm that recursively constructs the simple cones  $C^i, i \in I$ . In order to show that (c) holds, we just need to prove that all the  $n$  extreme rays of the cones constructed in such way have  $\{H, C\}$ -small complexity. Note that such extreme rays are either extreme rays of  $C$ , in which case have  $C$ -small complexity, or optimal solutions of a problem  $\min\{x^\top Hx : x \in F \cap \mathcal{H}\}$ , for a face  $F$  of  $C$ , in which case have  $\{H, C\}$ -small complexity.  $\square$   $\square$

Now we are ready to present a proof of Theorem 1.

*Proof of Theorem 1.* Consider a feasible  $\mathcal{F} = \mathcal{F}(H, c, d, A, b)$  and let  $Q(x) := x^\top Hx + c^\top x + d$  denote the quadratic form. Without loss of generality, we assume that the polyhedron  $\mathcal{C} := \{x : Ax \leq b\}$  is pointed: If not, consider the partition of the feasible region problem into  $2^n$  pieces as

$$\begin{aligned} x &\in \mathcal{F}(H, c, d, A, b) \\ x_i &\geq 0 \quad i \in S \subseteq \{1, \dots, n\} \\ x_i &\leq 0 \quad i \in \{1, \dots, n\} \setminus S, \end{aligned}$$

for every  $S \subseteq \{1, \dots, n\}$ ; note that the complexity of the additional constraints is  $O(n)$  and therefore each part in this partition has  $\mathcal{F}$ -small complexity, so we can restrict to a non-empty part.

Notice that an external description of  $\text{rec}(\mathcal{C})$  can be obtained by an external description of  $\mathcal{C}$  by replacing all the right-hand sides with a zero, hence  $\text{rec}(\mathcal{C})$  has  $\mathcal{F}$ -small complexity. Using our assumption that  $\text{rec}(\mathcal{C})$  is pointed, let  $\mathcal{H} := \{x : f^\top x = 1\}$  be the normalizing hyperplane from Lemma 6 for  $\text{rec}(\mathcal{C})$ . Examine the optimization problem:

$$\begin{aligned} \min \quad & r^\top Hr \\ \text{s.t.} \quad & r \in \text{rec}(\mathcal{C}) \cap \mathcal{H} \end{aligned} \quad (8)$$

Since  $\text{rec}(\mathcal{C}) \cap \mathcal{H}$  is compact, there exists a global optimal value. We break up into two cases depending on the sign of the optimal value.

**Case 1: The optimum for (8) is strictly negative.** We construct a feasible solution of  $\mathcal{F}(H, c, d, A, b)$  as follows. Since  $\text{rec}(\mathcal{C})$  and  $H$  have  $\mathcal{F}$ -small complexity, Theorem 3 asserts that there is an optimal solution  $r^*$  for (8) which has  $\mathcal{F}$ -small complexity. Then let  $\tilde{r}$  be an *integer* vector with  $\mathcal{F}$ -small complexity obtained by scaling  $r^*$ . Also, by Proposition 5, let  $\tilde{x}$  be a point in the mixed-integer linear set  $\mathcal{C} \cap (\mathbb{Z}^p \times \mathbb{R}^q)$  with  $\mathcal{F}$ -small complexity.

For every  $\lambda \in \mathbb{Z}_+$ , the point  $\tilde{x} + \lambda \tilde{r}$  belongs to  $\mathcal{C} \cap (\mathbb{Z}^p \times \mathbb{R}^q)$ . Expanding the quadratic form (and giving names to the different terms):

$$\mathcal{Q}(\tilde{x} + \lambda \tilde{r}) = \lambda^2 \tilde{r}^\top H \tilde{r} + \lambda (2\tilde{x}^\top H \tilde{r} + c^\top \tilde{r}) + c^\top \tilde{x} + d := \lambda^2 v_1 + \lambda v_2 + v_3. \quad (9)$$

Since  $v_1 < 0$ , this is a strictly concave polynomial in  $\lambda$ , and so setting  $\lambda$  larger than its larger root gives  $\mathcal{Q}(\tilde{x} + \lambda \tilde{r}) < 0$ . Explicitly, let

$$\tilde{\lambda} = \max \left\{ \left\lceil \frac{-v_2 - \sqrt{v_2^2 - 4v_1 v_3}}{2v_1} \right\rceil, 0 \right\}.$$

Then  $\tilde{x} + \tilde{\lambda} \tilde{r}$  is feasible for (2), and moreover it is easy to verify that it has  $\mathcal{F}$ -small complexity. This concludes the proof of Case 1.

**Case 2: The optimum for (8) is non-negative.** Then let  $\{P_i\}_i$  and  $\{R_K\}_{K \in \mathcal{K}}$  be the decomposition of  $\mathcal{C} \cap (\mathbb{Z}^p \times \mathbb{R}^q)$  from Proposition 4. By the guarantees of this decomposition, there is  $\bar{i}$  and  $\bar{K}$  such that  $(P_{\bar{i}} + \text{int.cone}(R_{\bar{K}})) \cap \{x : \mathcal{Q}(x) \leq 0\}$  is non-empty. Since  $R_{\bar{K}}$  is simple and pointed (since we assume  $\mathcal{C}$  pointed), we can use Lemma 7 to refine  $R_{\bar{K}}$  into the family of cones  $\{R_{\bar{K}, \bar{j}}\}_{\bar{j}}$ ; again due to its guarantees, let  $\bar{j}$  be such that  $(P_{\bar{i}} + \text{int.cone}(R_{\bar{K}, \bar{j}})) \cap \{x : \mathcal{Q}(x) \leq 0\}$  is non-empty. We show that this set has a point of  $\mathcal{F}$ -small complexity. To simplify the notation, let  $P := P_{\bar{i}}$ , and enumerate  $R_{\bar{K}, \bar{j}} = \{r^j\}_j$ .

Now let  $F := \text{cone}(r^j)_j$ . In addition, for an index  $i$ , we exclude ray  $r^i$  to define the face  $F_i := \text{cone}(r^j)_{j \neq i}$ , and similarly for a set of indices  $J$ , let  $F_J := \text{cone}(r^j)_{j \notin J}$ . Finally, we define the int.cone version of these cones, namely  $F^I := \text{int.cone}(r^j)_j$ ,  $F_i^I := \text{int.cone}(r^j)_{j \neq i}$  and  $F_J^I := \text{int.cone}(r^j)_{j \notin J}$ . So we are interested in the solutions to

$$\begin{aligned} \mathcal{Q}(x) &\leq 0 \\ x &\in P + F^I. \end{aligned} \quad (10)$$

We first analyze the behavior of  $\mathcal{Q}$  over a single direction  $r^j$ . Any point in  $P + F^I$  can be written as  $x^i + \mu r^i$  for  $x^i \in P + F_i^I$  and  $\mu \in \mathbb{Z}_+$ . Define  $J := \{j \in \{1, \dots, n\} : (r^j)^\top H r^j = 0\}$ . Then  $\mathcal{Q}$  has linear behavior along the directions  $r^i$  with  $i \in J$ : for all  $x^i \in P + F_i^I$  and  $\mu \in \mathbb{Z}_+$

$$\mathcal{Q}(x^i + \mu r^i) = \mu \cdot \left( 2(x^i)^\top H r^i + c^\top r^i \right) + (x^i)^\top H x^i + c^\top x^i + d \quad \forall i \in J. \quad (11)$$

Hence, if there is  $i \in J$  and a point  $x^i \in P + F_i^I$  such that the first term  $2(x^i)^\top H r^i + c^\top r^i$  is negative, then we can find a large scaling  $\mu$  such that the point  $x^i + \mu r^i$  satisfies (10); in fact, we can construct such a point in a way that it has  $\mathcal{F}$ -small complexity.

**Claim 1.** 1 Consider  $i \in J$  and the linear optimization problem  $\min\{2(x^i)^\top H r^i + c^\top r^i : x^i \in P + F_i^I\}$ . If the optimum of this problem is negative, then there is a point  $\tilde{x}^i \in P + F_i^I$  which has  $\mathcal{F}$ -small complexity and negative objective value.



*Proof of claim.* To simplify the notation, let  $obj(x) = 2x^\top Hr^i + c^\top r^i$  denote the objective function. Let  $\tilde{p} \in \operatorname{argmin}\{obj(p) : p \in P\}$ . Since  $P$  has  $\mathcal{F}$ -small complexity, it follows that  $\tilde{p}$  has  $\mathcal{F}$ -small complexity. Clearly if  $obj(\tilde{p}) < 0$ , then set  $\tilde{x}^i$  to  $\tilde{p}$  as the desired point in  $P + F_i^I$ , concluding the proof. Otherwise, by linearity of  $obj$  and the definition of  $F_i^I$  there exists some  $j \neq i$  such that  $2(r^j)^\top Hr^i < 0$ . Then let  $\tilde{\eta}_j$  be the smallest non-negative integer satisfying

$$\begin{aligned} obj\left(\tilde{p} + r^j \tilde{\eta}_j\right) &\leq -1 \\ \tilde{\eta}_j &\in \mathbb{Z}_+ \end{aligned}$$

Clearly  $\tilde{\eta}_j$  has  $\mathcal{F}$ -small complexity and therefore  $\tilde{x}^i = \tilde{p} + \tilde{\eta}_j r^j$  is the desired point in  $P + F_i^I$ , concluding the proof.  $\diamond$

Then suppose there is  $i \in J$  such that  $\min\{2(x^i)^\top Hr^i + c^\top r^i : x^i \in P + F_i^I\}$  is negative. From Claim 1, let  $\tilde{x}^i \in P + F_i^I$  have  $\mathcal{F}$ -small complexity such that  $\tilde{v} := 2(\tilde{x}^i)^\top Hr^i + c^\top r^i < 0$ ; notice that  $\tilde{v}$  has  $\mathcal{F}$ -small complexity. Given (11), we set  $\mu = \lceil \frac{(\tilde{x}^i)^\top H\tilde{x}^i + c^\top \tilde{x}^i + d}{|\tilde{v}|} \rceil$  to get that  $\tilde{x}^i + \mu r^i$  is feasible for the problem (10) and has  $\mathcal{F}$ -small complexity; this concludes the proof in this case.

Finally, consider the case where for all  $i \in J$  we have  $\min\{2(x^i)^\top Hr^i + c^\top r^i : x^i \in P + F_i^I\}$  non-negative. In this case, problem (10) is feasible if and only if

$$\begin{aligned} \mathcal{Q}(x) &\leq 0 \\ x &\in P + F_J^I \end{aligned} \tag{12}$$

is feasible.

First we bound the norm of solutions to the above problem.

**Claim 2.** *2 There is a rational number  $v^*$  of  $\mathcal{F}$ -small complexity such that for all  $x$  satisfying (12) we have  $\|x\| \leq v^*$ .*

*Proof of claim.* Let  $\mathcal{H} = \{x : f^\top x = 1\}$  be the normalizing hyperplane given by Lemma 6 for the cone  $F$ . Now consider any point of the form  $\bar{p} + \bar{r} \in P + F_J^I$  (with  $\bar{p} \in P$  and  $\bar{r} \in F_J^I$ ) such that  $\mathcal{Q}(\bar{p} + \bar{r}) \leq 0$ ; also consider the vector in direction  $\bar{r}$  that belongs to  $\mathcal{H}$ , namely let  $\tilde{r} = \lambda \bar{r}$  for  $\tilde{r} \in F_J \cap \mathcal{H}$  and  $\lambda > 0$ . We upper bound the norm of  $\bar{p} + \bar{r}$ , starting by bounding  $\lambda$ .

Since  $F$  satisfies the conclusion of Lemma 7, and given the definition of  $J$ , we have that  $r^\top Hr > 0$  for all  $r \in F_J$ . Let  $v_1^* = \min\{r^\top Hr : r \in F_J \cap \mathcal{H}\}$  (notice that  $F_J \cap \mathcal{H}$  is compact). Since  $F_J, \mathcal{H}$  and  $H$  have  $\mathcal{F}$ -small complexity, it follows from Theorem 3 that  $v_1^*$  also has  $\mathcal{F}$ -small complexity. Evaluating  $\mathcal{Q}$  over  $\bar{p} + \bar{r}$  we have

$$\mathcal{Q}(\bar{p} + \bar{r}) = \lambda^2 \tilde{r}^\top H \tilde{r} + \lambda \left( 2\tilde{r}^\top H \bar{p} + c^\top \tilde{r} \right) + \left( (\bar{p})^\top H \bar{p} + c^\top \bar{p} + d \right). \tag{13}$$

Let  $v_2^* := \min\{2r^\top Hp + c^\top r : p \in P, r \in F_J \cap \mathcal{H}\}$  and  $v_3^* := \min\{p^\top Hp + c^\top p + d : p \in P\}$ , so that  $\mathcal{Q}(\bar{p} + \bar{r}) \geq \lambda^2 v_1^* + \lambda v_2^* + v_3^*$ . Since  $v_1^* > 0$ , the polynomial  $\lambda^2 v_1^* + \lambda v_2^* + v_3^*$  is strictly convex (as a function of  $\lambda$ ), and since  $\mathcal{Q}(\bar{p} + \bar{r}) \leq 0$ , we have that  $\lambda$  cannot be larger than the largest of its roots; explicitly,  $\lambda \leq \left\lceil \frac{-v_2^* + \sqrt{(v_2^*)^2 - 4v_1^* v_3^*}}{2v_1^*} \right\rceil$ . Moreover, this bound is independent of our choice of point  $\bar{p} + \bar{r}$  and is a  $\mathcal{F}$ -small complexity value.



We can finally bound the norm of  $\bar{p} + \bar{r}$ . By triangle inequality,  $\|\bar{p} + \bar{r}\| \leq \|\bar{p}\| + \lambda\|\bar{r}\|$ . Let  $v_4^*$  be the ceiling of  $\|P\|_\infty$ .  $v_4^*$  has  $\mathcal{F}$ -small complexity, because it can be obtained as the ceiling of  $\max_i\{|\max\{(e^i)^\top p : p \in P\}|, |\min\{(e^i)^\top p : p \in P\}|\}$ . Therefore we can bound  $\|\bar{p}\| \leq \sqrt{n} \cdot v_4^*$ . Also, since  $\bar{r} \in \mathcal{H}$ , and by the definition of  $\mathcal{H}$  (see Lemma 6), we have  $f^\top \bar{r} = 1$  and  $f^\top \frac{\bar{r}}{\|\bar{r}\|} \geq \frac{1}{\max_j \|r^j\|}$ , which imply  $\|\bar{r}\| \leq \max_j \|r^j\|$ . Together, these bounds give an upper bound for  $\|\bar{p} + \bar{r}\|$  by a  $\mathcal{F}$ -small complexity value which is independent of  $\bar{p} + \bar{r}$ ; this concludes the proof.  $\diamond$

Now we show that if (12) has a feasible solution, then it has one of  $\mathcal{F}$ -small complexity. Let  $\bar{x}$  be a solution for (12) and recall that  $\bar{x} \in \mathbb{Z}^p \times \mathbb{R}^q$ . By the bound of Claim 2, and using integrality, we have that the first  $p$  components of  $\bar{x}$  have  $\mathcal{F}$ -small complexity. Then we fix these value and consider the optimization over the other components  $\min\{Q(x) : x \in P + F_J^I, x_i = \bar{x}_i \forall i \leq p\}$ . Claim 2 implies that this optimization problem has a global optimal solution and therefore from Theorem 3, we know that this optimization problem has an optimal solution  $\tilde{x}$  that has  $\mathcal{F}$ -small complexity, and by definition  $\tilde{x} \in P + F_J^I$  and  $Q(\tilde{x}) \leq Q(\bar{x}) \leq 0$ , and hence  $\tilde{x}$  is the desired feasible solution for (12). This concludes the proof.  $\square$   $\square$

## References

- [1] I. Borosh and L. B. Treybig, *Bounds on positive integral solutions to linear diophantine equations*, Proceedings of the American Mathematical Society **55** (1976), 299–304.
- [2] A. Del Pia and R. Weismantel, *Integer quadratic programming in the plane*, SODA (Chandra Chekuri, ed.), SIAM, 2014, pp. 840–846.
- [3] M.R. Garey, D.S. Johnson, and L. Stockmeyer, *Some simplified NP-complete graph problems*, Theoretical Computer Science **1** (1976), no. 3, 237–267.
- [4] R. Kannan and C. L. Monma, *On the computational complexity of integer programming problems*, Lecture Notes in Economics and Mathematical Systems (Rudolf Henn, Bernhard Korte, and Werner Oettli, eds.), vol. 157, Springer-Verlag, 1978, pp. 161–172.
- [5] R.M. Karp, *Reducibility among combinatorial problems*, Complexity of Computer Computations, Millera, R.E. and Thatcher, J.W., New York, 1972.
- [6] L. Khachiyan, *Convexity and complexity in polynomial programming*, Proceedings of the International Congress of Mathematicians **Warsaw** (1983).
- [7] M. Koeppe, *On the complexity of nonlinear mixed-integer optimization*, Mixed Integer Nonlinear Programming (Jon Lee and Sven Leyffer, eds.), The IMA Volumes in Mathematics and its Applications, vol. 154, Springer New York, 2012, pp. 533–557 (English).
- [8] J. C. Lagarias, *On the computational complexity of determining the solvability or unsolvability of the equation  $x^2 - dy^2 = -1$* , Transactions of the American Mathematical Society **260** (1980), 485–508.
- [9] C. H. Papadimitriou, *On the complexity of integer programming*, Journal of the Association for Computing Machinery **28** (1981).

- [10] A. Schrijver, *Theory of linear and integer programming*, John Wiley & Sons, Inc., New York, NY, USA, 1986.
- [11] S. A. Vavasis, *Quadratic programming is in NP*, Information Processing Letters **36** (1990), no. 2, 73–77.
- [12] J. von zur Gathen and M. Sieveking, *A bound on solutions of linear integer equalities and inequalities*, Proceedings of the American Mathematical Society **72** (1978), 155–158.